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CONTENTS

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A WAVE OPTICS CALCULATION OF PULSED LASER PROPAGATION IN GASES

INTRODUCTION

A theoretical description of the thermal blooming* of pulsed, focused laser beams has been improved by using the equations of wave optics. Previous approximations of the actual situation have used a geometrical optics formulation to describe both a collimated beam (1,2) and a focused Gaussian beam (3) in which focusing has been adjoined in an ad hoc fashion. The present study also serves to check the validity of the previous work.

The problem is nonlinear and numerical methods must be utilized (4) to study many cases of interest. Linearization restricts the range of validity but yields parametric expressions which can aid in understanding and collating the numerical results. Thus, a perturbation theory is used to study the effect of the atmospheric heating and the concomitant index of refraction change on the unperturbed (vacuum) wave amplitude. The heat source is taken to be proportional to the unperturbed wave. This result represents the first term of a self-consistent iteration scheme.

DERIVATION OF LASER FIELD EQUATIONS

Maxwell's equations give the wave equation for the electric vector, E:

$$\nabla(\nabla \cdot \mathrm{E}) - \nabla^2 \mathrm{E} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\epsilon \mathrm{E}), \tag{1}$$

where ϵ is the dielectric constant. Since depolarization is negligible for all cases of interest, Eq. (1) reduces to the scalar wave equation,

$$\nabla^2 \mathcal{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\epsilon \mathcal{E}) = 0, \tag{2}$$

where $\mathcal{E}$ is, for example, one component of linear polarization of the wave vector. Let $\mathcal{E} = \Psi e^{-i\omega t}$, where Ψ is assumed to be slowly varying in time compared with $1/\omega$. Then Eq. (2) becomes

*When a laser beam passes through a gas, energy is extracted from the beam and the gas is heated. If the product of beam intensity times the linear absorption coefficient of the gas is sufficiently large, the resultant density changes represent index-of-refraction changes sufficient to measurably alter the beam propagation. This effect is known as "thermal blooming" or "thermal lensing."

$$\nabla^2\Psi + \frac{\omega^2}{c^2}\,\Psi + \frac{2i\omega}{c^2}\,\frac{\partial\epsilon}{\partial t}\,\Psi - \frac{2}{c^2}\,\frac{\partial\epsilon}{\partial t}\,\frac{\partial\Psi}{\partial t} + \frac{2i\omega\epsilon}{c^2}\,\frac{\partial\Psi}{\partial t} = 0, \tag{3}$$

where $\partial^2\epsilon/\partial t^2$ and $\partial^2\Psi/\partial t^2$ have been dropped. This is a good approximation since the variation in time of the beam amplitude is due to the time variation of the index of refraction. These changes depend on fluid motions and are characteristically on the order of the beam diameter divided by the sound velocity, i.e., much longer than an optical period.

The beams to be considered have focal lengths much in excess of their diameters so that the paraxial approximation (5) is appropriate. Let $\Psi = \Phi e^{ik\sqrt{\epsilon_0}\,z}$, where Φ is slowly varying in z. The paraxial character of the wave makes $|\partial^2\varphi/\partial z^2| << |2ik(\partial\varphi/\partial z)|$ so that the former term may be neglected, resulting in

$$\nabla_\perp^2\Phi + 2ik\,\frac{\partial\Phi}{\partial z} + \left[k^2(\epsilon-\epsilon_0) + 2ik\,\frac{\partial\epsilon}{\partial t}\right]\Phi + \left(2i\,\frac{k\epsilon}{c} - \frac{2}{c^2}\,\frac{\partial\epsilon}{\partial t}\right)\frac{\partial\Phi}{\partial t} = 0, \tag{4}$$

where $k = \omega/c$.

The field amplitude is now linearized by making the decomposition

$$\Phi = \Phi_0 + \Phi_1, \tag{5}$$

where Φ_0 describes time-dependent propagation in vacuum and Φ_1 is a small perturbation due to heating of the gas which has sustained passage of a beam with amplitude Φ_0.

Furthermore, the dielectric constant is altered by a first-order correction,

$$\epsilon = \epsilon_0 + \epsilon_1, \tag{6}$$

where ϵ_1 is the perturbation of the ambient value ϵ_0 due to heating by a beam of amplitude Φ_0.

These equations lead to a zero- and first-order approximation to the problem. The zero-order equation is

$$\nabla_\perp^2\Phi_0 + 2ik\sqrt{\epsilon_0}\,\frac{\partial\Phi_0}{\partial z} + 2ik\,\frac{\epsilon_0}{c}\,\frac{\partial\Phi_0}{\partial t} = 0, \tag{7}$$

while the first-order equation is

$$\begin{aligned}\nabla_\perp^2\Phi_1 &+ 2ik\sqrt{\epsilon_0}\,\frac{\partial\Phi_1}{\partial z} + \left(k^2\epsilon_1 + 2ik\,\frac{1}{c}\,\frac{\partial\epsilon_1}{\partial t}\right)\Phi_0 \\ &+ \frac{2ik}{c}\,\epsilon_0\,\frac{\partial\Phi_1}{\partial t} + \left(\frac{2ik}{c}\,\epsilon_1 - \frac{2}{c^2}\,\frac{\partial\epsilon_1}{\partial t}\right)\frac{\partial\Phi_0}{\partial t} = 0.\end{aligned} \tag{8}$$

This result can be simplified further by use of the following inequalities which are extremely well obeyed with the possible exception of beams with extremely fast rise time or extremely short durations:

$$|k^2 \epsilon_1 \Phi_0| >> \left| \frac{2ik\epsilon_1}{c} \frac{\partial \Phi_0}{\partial t} \right| \tag{9}$$

$$\left| \frac{2ik}{c} \frac{\partial \epsilon_1}{\partial t} \Phi_0 \right| >> \left| \frac{2}{c^2} \frac{\partial \epsilon_1}{\partial t} \frac{\partial \Phi_0}{\partial t} \right|. \tag{10}$$

In addition the following condition holds since the density responds to the heating in times on the order of the hydrodynamic time (beam radius/sound speed):

$$|k^2 \epsilon_1| >> \left| \frac{2ik}{c} \frac{\partial \epsilon_1}{\partial t} \right|. \tag{11}$$

Finally then, the equation describing the propagation of the perturbed wave, accurate to first order, is

$$\nabla_\perp^2 \Phi_1 + 2ik \frac{\partial \Phi_1}{\partial z} + \frac{2ik}{c} \frac{\partial \Phi_1}{\partial t} = -k^2 3N\rho_1(\Phi_0)\Phi_0, \tag{12}$$

where the Lorentz-Lorenz law relating dielectric constant to gas density change has been used:

$$\epsilon_1 = 3N\rho_1. \tag{13}$$

SOLUTION FOR UNPERTURBED FIELD

In this section Eq. (7) is solved along with the initial condition

$$\Phi_0(\mathrm{x,y,z,}0) = 0 \tag{14}$$

and boundary conditions

$$\Phi_0(x,y,0,t) = f(x,y,t)$$

$$\Phi_0(\infty,\infty,z,t) = 0$$

$$\frac{\partial \Phi_0}{\partial x}(\infty,\infty,z,t) = \frac{\partial \Phi_0}{\partial y}(\infty,\infty,z,t) = 0. \tag{15}$$

The amplitude is Fourier-analyzed

$$V(\xi,\eta,z,t) = \iint_{-\infty}^{\infty} e^{i\xi z} e^{i\eta y} \Phi_0(x,y,z,t)\, dxdy$$

so that

$$\frac{\partial V}{\partial z} + \frac{1}{c}\frac{\partial V}{\partial t} + \frac{i}{2k}(\xi^2 + \eta^2) V = 0. \tag{16}$$

A Laplace transformation of the time variable is applied:

$$W(\xi,\eta,z,s) = \int_0^\infty e^{-st}\, V(\xi,\eta,z,t)\, dt \tag{17}$$

giving

$$\frac{\partial W}{\partial z} + \frac{i}{2k}\, (\xi^2 + \eta^2) W + \frac{s}{c}\, W = 0, \tag{18}$$

where the initial condition in Eq. (14) has been used. Equation (18) is solved by

$$W = \exp\left(-\frac{i}{2k}\, (\xi^2 + \eta^2)\, z - \frac{sz}{c}\right) A(\xi,\eta,s).$$

To find $A(\xi,\eta,s)$, W is evaluated at the plane $z = 0$,

$$\begin{aligned} A(\xi,\eta,s) &= W(\xi,\eta,0,s) \\ &= \int_0^\infty e^{-st}\, V(\xi,\eta,0,t)\, dt \\ &= \frac{1}{2\pi} \int_0^\infty e^{-st}\, dt \iint_{-\infty}^{\infty} e^{i\xi n}\, e^{i\eta v} f(u,v,t)\, du dv. \end{aligned}$$

Thus,

$$\begin{aligned} V(\xi,\eta,z,t) = &\int_{-i\infty+\gamma}^{i\infty+\gamma} e^{st}[e^{(-i/2k)(\xi^2+\eta^2)z}\, e^{-sz/c}]\, ds \int_0^\infty e^{-st'}\, dt' \\ &\times \iint_{-\infty}^{\infty} f(u,v,t')\, e^{i\xi u}\, e^{i\eta v}\, du dv; \end{aligned} \tag{19}$$

hence,

$$\begin{aligned} \Phi_0(x,y,z,t) = &\frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d\xi d\eta \int_{-i\infty+\gamma}^{i\infty+\gamma} ds \int_0^\infty dt' \iint_{-\infty}^{\infty} f(u,v,t') \\ &\times \exp\left[st - \frac{i}{2k}\, (\xi^2 + \eta^2)z - \frac{sz}{c} - st' + i\xi(u-x) + i\eta(v-y)\right] du dv. \end{aligned} \tag{20}$$

Now, for any function $g(t)$,

$$\int_{-i\infty+\gamma}^{i\infty+\gamma} e^{st}\, ds \left[e^{-sz/c} \int_0^{\infty} e^{-t's} g(t')\, dt' \right] = g(t - z/c); \tag{21}$$

in addition,

$$\iint_{-\infty}^{\infty} f(u,v,t-z/c)\, du dv \iint_{-\infty}^{\infty} e^{i\xi(u-x)}\, e^{i\eta(v-y)}\, e^{i(\xi^2 + \eta^2)z/2k}\, d\xi d\eta \tag{22}$$

is just the solution of the steady-state problem, translated in time by $t - (z/c)$. For example, if the amplitude on the initial plane is turned on at some time t and is time independent thereafter, the amplitude at z is zero until $t + (z/c)$, and is just the usual time-independent amplitude from that time on.

Only pulsed Gaussian focused beams at the source are considered. In this case the zero-order field (5) *is*

$$\Phi_0(x,y,z,t) = \frac{-i}{\sqrt{\pi a^2}\left[\frac{z}{ka^2} - i\left(1 - \frac{z}{f}\right)\right]} \exp\left\{ - \frac{x^2 + y^2}{2a^2\left[\left(\frac{z}{ka^2}\right)^2 + \left(1 - \frac{z}{f}\right)^2\right]} + i\,\frac{x^2 + y^2}{2a^2 z}\left[1 - \left(\frac{z}{ka^2}\right)^2 + \left(1 - \frac{z}{f}\right)^2\right]\right\} \theta\left(t - \frac{z}{c} - t_1\right)\theta\left(t_2 - t + \frac{z}{c}\right), \tag{23}$$

where

$$\theta(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

and t_1, t_2 are pulse turn-on and turn-off times, respectively.

FIRST ORDER FIELD CALCULATION

In this section Eq. (12) is solved, under the conditions

$$\Phi_1(x,y,0,t) = 0$$

$$\Phi_1(\infty,\infty,z,t) = \frac{\partial \Phi_1}{\partial x}(\infty,\infty,z,t) = \frac{\partial \Phi_1}{\partial y}(\infty,\infty,z,t) = 0$$

$$\Phi_1(x,y,z,0) = 0. \tag{24}$$

Define

$$F(x,y,z,t) = 3k^2 N \rho_1(\Phi_0)\Phi_0(x,y,z,t). \tag{25}$$

The method of Duhamel (6) is employed to solve the inhomogeneous initial value problem. Let $v(x,y,z,t;\tau)$ be a one-parameter family of solutions to the homogeneous problem

$$\nabla_{\perp}^2 v + 2ik \frac{\partial v}{\partial z} + \frac{2ik}{c} \frac{\partial v}{\partial t} = 0$$

subject to the initial condition

$$v(x,y,z,0;\tau) = \frac{c}{2ik} F(x,y,z,\tau).$$

Then,

$$\Phi_1(x,y,z,t) = \int_0^t v(x,y,z,t-\tau;\tau)\, d\tau. \tag{26}$$

The proof follows directly by substitution in the inhomogeneous equation.

Duhamel's principle reduces the problem to a solution of a homogeneous equation, which, unlike the problem in the previous section, has a nonzero initial condition in place of a nonzero boundary condition.

Fourier analysis in x and y is again applied:

$$V(\xi,\eta,z,t) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} e^{i\xi x}\, e^{i\eta y}\, v(x,y,z,t;\tau)\, dxdy, \tag{27}$$

but here it is followed by Laplace transformation in z instead of t due to the new boundary condition:

$$W(\xi,\eta,s,t) = \int_0^{\infty} e^{-sz'} V(\xi,\eta,z',t)\, dz'. \tag{28}$$

This results in

$$W(\xi,\eta,s,t) = A(\xi,\eta,s) \exp\left[-\frac{i}{2k} (\xi^2 + \eta^2)ct - sct\right] \tag{29}$$

with $A(\xi,\eta,s)$ to be determined. Let $t = 0$; then

$$\begin{aligned} A(\xi,\eta,s) &= W(\xi,\eta,s,0) \\ &= \int_0^{\infty} e^{-sz'} V(\xi,\eta,z',0)\, dz' \\ &= \frac{1}{2\pi} \int_0^{\infty} e^{-sz'}\, dz' \iint_{-\infty}^{\infty} e^{i\xi x}\, e^{i\eta y} \frac{c}{2ik} F(x,y,z',\tau)\, dxdy. \end{aligned} \tag{30}$$

Thus,

$$W(\xi,\eta,s,t) = e^{(-i/2k)(\xi^2+\eta^2)ct} e^{-sct}$$

$$\times \int_0^\infty \left\{ \frac{e^{-sz'}}{2\pi} \iint_{-\infty}^{\infty} \left[e^{i\xi x}\, e^{i\eta y}\, \frac{c}{2ik}\, F(x,y,z',\tau) \right] dxdy \right\} dz'. \qquad (31)$$

Now inverting the Laplace and Fourier transforms gives

$$\Phi_1(x,y,z,t) = \frac{1}{2\pi} \int_0^t d\tau \iint_{-\infty}^{\infty} d\xi d\eta\, e^{-\xi x}\, e^{-i\eta y}$$

$$\times \int_{-i\infty+\gamma}^{i\infty+\gamma} ds\, e^{sz}\, e^{(-i/2k)(\xi^2+\eta^2)c(t-\tau)-sc(t-\tau)}$$

$$\times \int_0^\infty \left[\frac{e^{sz'}}{2\pi} \iint_{-\infty}^{\infty} e^{i\xi u}\, e^{i\eta v}\, \frac{c}{2ik}\, F(u,v,z',\tau)\, dudv \right] dz'. \qquad (32)$$

Furthermore,

$$\int_{-\infty+i\gamma}^{\infty+i\gamma} ds\, e^{sz}\, e^{-sc(t-\tau)} \int_0^\infty e^{-sz'}\, F(u,v,z',\tau)\, dz' = F(u,v,z-c(t-\tau),\tau) \qquad (33)$$

and also,

$$\left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} e^{-i\xi(x-u)}\, e^{-i\eta(y-v)}\, e^{(-i/2k)(\xi^2+\eta^2)c(t-\tau)}\, d\xi d\eta$$

$$= \frac{-ik}{2\pi c(t-\tau)} \exp\left\{ \frac{ik}{2\tau(t-\tau)} \left[(u-x)^2 + (v-y)^2 \right] \right\}. \qquad (34)$$

Finally,

$$\Phi_1(x,y,z,t) = -\frac{1}{4\pi} \int_0^t d\tau\, \frac{1}{(t-\tau)} \iint_{-\infty}^{\infty} F(u,v,z-c(t-\tau),\tau)$$

$$\times \exp\left\{ \frac{ik}{2c(t-\tau)} \left[(u-x)^2 + (v-y)^2 \right] \right\} dudv. \qquad (35)$$

It will be more convenient to transform the integration in τ to an integration over range. Let $\zeta = z - c(t-\tau)$. Then Eq. (35) becomes

$$\Phi(x,y,z,t) = -\frac{1}{4\pi}\int_0^z d\zeta \, \frac{1}{z-\zeta}$$

$$\times \iint_{-\infty}^{\infty} F(u,v,\zeta,t-(z-\zeta)/c) \exp\left\{\frac{ik}{2(z\ \zeta)}\left[(ux)^2 + (v-y)^2\right]\right\} du dv, \tag{36}$$

where the lower end point is 0 instead of $z - ct$ since $F(x,y,\zeta,t-(z-\zeta)/c)$ is zero both for $\zeta < 0$ and for $z > ct$. In the previous section we saw that for a beam which is turned on at $t = 0$ at the laser face and is time independent thereafter, the beam downrange in vacuum is just delayed by the light propagation time z/c but is otherwise the steady-state beam. Thus, we can replace $F(x,y,\zeta,t-(z-\zeta)/c$ by $F(x,y,\zeta,t)$ since $F(x,y,\tau,t)$ is a function of the vacuum beam alone. Time at any range is now to be counted from the time the beam arrives at that range. With this change Eq. (36) is just the solution of Eq. (12) with the term $(2ik/c)\ \partial\Phi_1/\partial t$ dropped. This fact is demonstrated in the appendix and describes the situation where the beam changes are due solely to the relatively sluggish response of the gas to beam heating.

PERTURBED FIELD IN RESPONSE TO A LONG PULSE

Equations (36), (25), and (23) can now be combined with an expression for the density change (1) in first order in the limit of pulses which are long in comparison with hydrodynamic times:

$$\rho_1(x,y,z) = -\frac{(\gamma-1)\alpha t_p}{c_s^2}\ |\Phi_0|^2, \tag{37}$$

where γ is the ratio of specific heats, α is the linear absorption coefficient, and t_p is the pulse length. Thus,

$$F(u,v,\zeta,\tau) = \frac{3Nk^2(\gamma-1)\alpha t_p}{c_s^2}\ \frac{P}{(\pi a^2)^{3/2}}\ \frac{-i}{D^2(\zeta)}\ (\zeta/ka^2 + i - \zeta/f)$$

$$\times \exp\left\{\left[-\frac{3}{2a^2 D(\zeta)} - \frac{ik}{2\zeta}\left(1 - \frac{1-\zeta/f}{D(\zeta)}\right)\right](u^2+v^2)\right\}, \tag{38}$$

where P is the total power at the laser face,

$$D(\zeta) = \left(\frac{\zeta}{ka^2}\right)^2 + (1-\zeta/f)^2 \tag{39}$$

and f is the focal length. Let $\bar{u} = u/a$, $\bar{v} = v/a$, and evaluate Eq. (36) on axis so that

$$\Phi_1(0,0,z) = c_1 \int_0^z d\zeta\ \frac{\dfrac{\zeta}{ka^2} + i(1-\zeta/f)}{D^2(\zeta)(z-\zeta)} \tag{40}$$

Continued

$$\times \iint_{-\infty}^{\infty} \exp \left\{ (\bar{u}^2 + \bar{v}^2) \left(-\frac{3}{2D(\zeta)} \right. \right.$$

$$\left. \left. + i \left(1 - \frac{1 - \zeta/f}{D(\zeta)} \right) \frac{ka^2}{2\zeta} + \frac{ika^2}{2(z-\zeta)} \right) \right\} d\bar{u}\, d\bar{v}, \tag{40}$$

where

$$C_1 = \frac{3iNk^2(\gamma - 1)\,\alpha P t_p}{4c_s^2 a \pi^{5/2}}. \tag{41}$$

The integrations over $\bar{u}$ and $\bar{v}$ can be readily performed to give

$$\Phi_1(0,0,z) = \pi c_1 \int_0^z \frac{\zeta/ka^2 + i(1 - \zeta/f)}{D^2(\zeta)(z-\zeta) \left\{ \frac{3}{2D(\zeta)} - i \left[\frac{ka^2}{2(z-\zeta)} + \frac{ka^2}{2\zeta} \left(1 - \frac{1-\zeta/f}{D(\zeta)} \right) \right] \right\}} d\zeta. \tag{42}$$

A quantity of interest is the ratio of the intensity of the perturbed beam on axis at the focal point to the intensity of the unperturbed beam there. Equation (42) is therefore evaluated at $z = f$ to give

$$\Phi_1(0,0,f) = \frac{\pi c_1}{f} \int_0^f \frac{\zeta/ka^2 + i(1-\zeta/f)}{D(\zeta)(1-\zeta/f) \left[\frac{3}{2} - i \frac{\zeta/ka^2}{2(1-\zeta/f)} \right]} d\zeta. \tag{43}$$

Let

$$x = \frac{1 - \zeta/f}{\zeta/ka^2}.$$

Then

$$\Phi_1(0,0,f) = \frac{2\pi c_1 ka^2}{f} \int_0^\infty \frac{(1+ix)(3x+i)}{(1+x^2)(9x^2+1)} dx. \tag{44}$$

Only the real part of the integrand is kept since

$$\frac{|\Phi_0 + \Phi_1|^2}{|\Phi_0|^2} = 1 + 2 \operatorname{Re} \left(\Phi_0 \Phi_1^* \right) = 1 + 2 \operatorname{Im} \left(\Phi_0 \right) \operatorname{Im} \left(\Phi_1 \right) \tag{45}$$

to first order in the perturbed field and both c_1 and Φ_0 are pure imaginary. Thus,

$$\Phi_1(0,0,f) = 2\pi c_1 \frac{ka^2}{f} \int_0^\infty \frac{2x\,dx}{(1+x^2)(9x^2+1)}$$

$$= \frac{2\pi c_1}{f} ka^2 \frac{\log 9}{8}, \tag{46}$$

so that

$$\Phi_1(0,0,f) = \frac{3i \log 9N(\gamma-1)\alpha t_p P}{16c_s^2\,\pi\sqrt{\pi}}\,\frac{k^3 a}{f}. \tag{47}$$

Finally, the ratio of perturbed-to-unperturbed intensity on axis at the focal point is

$$\frac{|\Phi_0 + \Phi_1|^2}{|\Phi_0|^2} = 1 - \frac{\log 9}{4\pi}\,\frac{3N}{2}\,\frac{(\gamma-1)\alpha E k^2}{c_s^2}, \tag{48}$$

where E is the total energy delivered in the time t_p. It is interesting to note that this result is independent of focal length f and beam size a.

PERTURBED FIELD IN RESPONSE TO A SHORT PULSE

The density change induced in times short compared with a hydrodynamic time scales is given (1) by

$$\rho_1 = \frac{(\gamma-1)\alpha t_p^3 \nabla^2 |\Phi_0|^2}{6}. \tag{49}$$

The steps to solution are exactly as in the previous section. The last quadrature is not completed exactly, however. The analogous equation to Eq. (46) for this case is

$$\Phi_1(0,0,f) = 2\pi c_2 \frac{ka^2}{f} \int_0^\infty 2x \frac{(x + ka^2/f)(6x^2+2)}{(1+x^2)^2 (9x^2+1)^2}\,dx, \tag{50}$$

where

$$c_2 = -\frac{iNk^2(\gamma-1)\alpha t_p^3 P}{2\pi^{5/2} a^3}. \tag{51}$$

Now ka^2/f is typically 10 to 100 for problems of interest. Thus, when $x \approx ka^2/f$, the integral has pretty well converged due to the dominance of powers of x in the denominator. For $x << ka^2/f$, the quantity ka^2/f can be factored out of the integrand to give

$$\Phi_1(1,0,f) \approx 2\pi c_2 \left(\frac{ka^2}{f}\right)^3 \int_0^\infty \frac{2x(6x^2+2)}{(1+x^2)^2(9x^2+1)^2}\,dx \tag{52}$$

Continued

$$= -2i\left(\frac{3}{128}\log 9 + \frac{1}{16}\right)\frac{N(\gamma-1)\alpha t_p{}^2 E}{\pi\sqrt{\pi}}\frac{k^5a^3}{f^3}.$$

Thus,

$$\frac{|\Phi_0 + \Phi_1|^2}{|\Phi_0|^2} = 1 - \left(\frac{1}{16}\log 9 + \frac{1}{6}\right)\frac{3N}{2}\frac{(\gamma-1)\alpha t_p{}^2 E}{\pi}\frac{k^4a^2}{f^2}. \tag{53}$$

A geometrical optics calculation of the on-axis intensity ratio when focusing is adjoined (3,7) gives, for the short pulse case,

$$\frac{I}{I_0} = 1 - \delta_0\left[\frac{3}{2\pi}N(\gamma-1)\alpha E t_p{}^2\right]\frac{k^4a^2}{f^2}, \tag{54}$$

where δ_0 is a number like 0.4. This should be compared with Eq. (53). The long pulse result in geometrical optics (3,7) is

$$\frac{I}{I_0} = 1 - \delta_\infty\left[\frac{3}{2\pi}N(\gamma-1)\frac{\alpha k^2}{c_s^2}E\right],$$

where δ_∞ is a number of order unity. This should be compared with Eq. (48).

Thus, the results differ in both cases only by a numerical factor. This lends credence to the methods of Ref. (3) to incorporate focusing in a geometric optics formulation of beam propagation.

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Appendix A

SOLUTION FOR INFINITE LIGHT VELOCITY

This appendix demonstrates that Eq. (36) is the solution of Eq. (12) with $(2ik/c)$ $\partial\Phi_1/\partial t$ set equal to zero. The equation to be solved is

$$\nabla_\perp^2\Phi_1 + 2ik \frac{\partial\Phi_1}{\partial z} = F(x,y,z,t). \tag{A1}$$

Duhamel's principle is again employed. Let $v(x,y,z;\zeta)$ be a one-parameter family of solutions to the homogeneous problem

$$\nabla_\perp^2 v + 2ik \frac{\partial v}{\partial z} = 0 \tag{A2}$$

subject to the condition

$$v(x,y,0;\zeta) = F(x,y,\zeta,t)/2ik. \tag{A3}$$

Then,

$$\Phi_1(x,y,z) = \int_0^z v(x,y,z-\zeta;\zeta)\, d\zeta. \tag{A4}$$

The homogeneous problem is solved by Fourier analysis to give for $V(\xi,\eta,z)$, the Fourier transform of $v(x,y,z;\zeta)$,

$$V(\xi,\eta,z) = A(\xi,\eta)e^{(-i/2k)(\xi^2+\eta^2)z}, \tag{A5}$$

where

$$A(\xi,\eta) = V(\xi,\eta,0) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} e^{i\xi x}\, e^{i\eta y}\, v(x,y,0;\zeta)\, dxdy. \tag{A6}$$

Inverting the Fourier integrals gives

$$v(x,y,z;\zeta) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} dudv \iint_{-\infty}^{\infty} \frac{F}{4\pi ik}\, e^{(-i/2k)(\xi^2+\eta^2)z}\, e^{i\xi(u-x)}\, e^{i\eta(v-y)}\, d\xi d\eta. \tag{A7}$$

Substituting Eq. (A7) into Eq. (A4) and integrating with respect to ξ and η gives

$$\Phi_1(x,y,z,t) = \frac{1}{4\pi} \int_0^z \frac{1}{(z-\zeta)}\, d\zeta \iint_{-\infty}^{\infty} F(u,v,\zeta,t)$$

$$\times \exp\left\{\frac{ik}{2(z-\zeta)} \left[(u-x)^2 + (v-y)^2\right]\right\} du dv, \qquad \text{(A8)}$$

which except for the time dependence is exactly Eq. (36). If time is measured at each range beginning at the arrival of the light beam at that range, then Eqs. (21) and (22) imply that Eq. (A8) and Eq. (36) are equivalent.